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Inertial range scaling of intense events in turbulence

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The inertial range scaling exponents $\zeta(p)$ for the velocity difference moments $\langle[u(x+r)-u(x)]^p\rangle$ in high Reynolds number turbulence are considered for large values of p. Two different models are considered. In the first the asymptotic behavior of $\zeta(p)$ is $1.440p^{1/2}$, and in the second it is $(p/9)$. It is shown that these two models are indistinguishable experimentally for $p<100$, but that the underlying dynamical assumption about the most intense turbulent events in the two models is essentially different.

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The velocity structure functions $\langle[u(x+r)-u(x)]^p\rangle$ in the inertial range of high Reynolds number incompressible turbulence are usually assumed to scale as $r^{\zeta(p)}$. In the 1941 Kolmogorov theory [13], the probability density function (PDF) for $[u(x+r)-u(x)]$ is self-similar, and the scaling exponents $\zeta(p)=p/3$. Experimentally and in direct numerical simulations, these scaling exponents increase less rapidly with p. This problem of the internal intermittency of turbulent flows has generated a large literature which I have recently reviewed [1]. There are many models for the dependence of $\zeta(p)$ on p. In this note I concentrate on the behavior of $\zeta(p)$ for large values of p. I make the strong assumption that all high order moments exist, and that power law scaling is applicable to them. These high order moments then describe the scaling of the most intense inertial range events. I emphasize the physical assumptions underlying contrasting models for the scaling behavior of these intense events.

It is usually assumed that inertial range scaling is governed by the Kolmogorov refined similarity hypothesis [2] which states that

$$\zeta(p)=(p/3)-\mu(p/3), \tag{1}$$

where $\mu(p)$ is defined by

$$\langle\varepsilon_r^p\rangle=\text{const}\times r^{-\mu(p)}, \tag{2}$$

and ε_r is the local dissipation averaged over an inertial range distance r. In this note, I will assume the refined similarity hypothesis to apply.

In a recent paper Novikov [3] has applied the theory of infinitely divisible distributions to the scaling properties of ε_r, making essential use of the fact that ε_r is a non-negative random variable. In particular, he focuses attention on a parameter h defined by

$$h=\lim_{p\to\infty}[\mu(p)/p]. \tag{3}$$

He emphasizes that if $h<1$, then there is a gap in the PDF for the ratio variable $(\varepsilon_r/\varepsilon_{br})$, with $b<1$. He points out that a recent model by She and Leveque [4] gives $h=2/3$, and thus implies a substantial gap in this PDF. He suggests that this gap is unphysical and in contradiction to experiment. In the present note, I consider the physical implications of the value of h. I suggest that there is an interesting unresolved question of principle, but that the question is unlikely to be resolved by experiment.

Using Eq. (1), I first note that refined similarity implies that

$$(1-h)=3\lim_{p\to\infty}[\zeta(p)/p)], \tag{4}$$

so that if $h=1$, the exponents $\zeta(p)$ increase less rapidly than linearly for large p. When $h=1$, the dominant linear term in $\mu(p)$ for large p exactly cancels the $(p/3)$ in Eq. (1). In this case, the subdominant term in $\mu(p)$ determines the asymptotic behavior of $\zeta(p)$. In terms of the usual multifractal formalism for the dissipation fluctuations [5], the scaling properties of ε_r are described in terms of a function $f(\alpha)$ where α extends over a range from $\alpha_{\min}$ corresponding to the most intense events to $\alpha_{\max}$ corresponding to the least intense events. If $h=1$, then $\alpha_{\min}=0$, while if $h<1$, then $\alpha_{\min}>0$.

A reasonable choice for the asymptotic behavior of $\zeta(p)$ is p^γ, with $\gamma<1$, but how can the exponent γ be estimated? One possibility is suggested by a dynamical model for a passive scalar advected by a white noise Gaussian velocity field with a Kolmogorov spatial spectrum. This has been studied by Kraichnan [6], and the results have recently been confirmed and extended by Kraichnan, Yakhot, and Chen [7]. In this model system, the scaling exponents $\zeta(p)$ are proportional to $p^{1/2}$ for large p. There is no good reason to assume that this asymptotic $p^{1/2}$ behavior will be valid for the nonlinear dynamics of the Navier-Stokes equations, but it is one possibility.

To be more specific, consider a family of analytical models discussed by Novikov as Eq. (18) in Ref. [3] which contains three parameters κ, γ, and σ. (I use the symbol γ instead of Novikov's α to avoid confusion with the usual multifractal formalism.) If $h=1$, then $\kappa=1$. Following Kraichnan's passive scalar model, I take $\gamma=1/2$. Novikov's Eq. (18) for $\mu(p)$, using Eq. (1), then gives

$$\zeta_N(p)=\{[1+(p\sigma/3)]^{1/2}-1\}/\{(1+\sigma)^{1/2}-1\}. \tag{5}$$

To choose the parameter σ, note that $\zeta(6)=2-\mu(2)=2-\mu$, where μ is the putatively universal exponent defined by the spectrum of dissipation fluctuations. I want to compare Eq.

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TABLE I. Comparison of the scaling exponents $\zeta(p)$ from the She-Leveque model of Eq. (6) and the present model, Eq. (5) with $\sigma=0.9126$.

Order p	She-Leveque	Eq. (5)
2	0.696	0.700
3	1.000	1.000
4	1.280	1.277
6	1.778	1.778
8	2.211	2.227
10	2.593	2.639
20	4.088	4.339
40	6.435	6.864
60	8.666	8.846
80	10.889	10.532
100	13.111	12.025
200	24.222	17.923

(5) with the model of She and Leveque, which gives $\mu=2/9$, in agreement with most experiments [8]. It is easily verified that Eq. (5) gives $\zeta(6)=1.7778$ if the parameter $\sigma=0.9126$. In Eq. (5), the subscript N denotes any name that the reader would like to use.

In particular, compare Eq. (5) to the model proposed by She and Leveque in Ref. [4] which gives

$$\zeta_{\rm SL}(p)=(p/9)+2[1-(2/3)^{p/3}]. \tag{6}$$

Equations (5) and (6) are constructed to give $\zeta(3)=1$, as required by the Navier-Stokes equations, and discussed, for example, in Ref. [1]. The choice of the parameter $\sigma=0.9126$ forces the two models to also agree for $\zeta(6)=2-\mu=16/9$. They behave very differently for large p, however, with $\zeta_{\rm SL}(p)$ asymptotically approaching $p/9$, while $\zeta_N(p)$ asymptotically approaches $1.440p^{1/2}$. I had expected that the two models would differ markedly in their functional form over a reasonable range of p. The predictions of the two models are compared in Table I for a range of p extending from 2 to 200. This gives the surprising result that the difference between the two models is negligible for $p<100$ despite the fact that their asymptotic behavior for large p is very different. It is totally unfeasible to distinguish between the two models experimentally at the level of scaling exponents for the moments.

What is the essential physical difference between the two models, and where does the gap in the She-Leveque model originate? As shown in Ref. [4], the essential feature is the dependence of the most intense events on scale size. This is described by a quantity $\varepsilon_r^{(\infty)}$ which scales as $r^{-2/3}$ in the She-Leveque model. In this model, $\varepsilon_r^{(\infty)}=U^2/\tau(r)$, where U is a large scale velocity, and $\tau(r)$ is the Kolmogorov time scale for an eddy of size r. This leads directly to $h=2/3$ in Novikov's language. To obtain $h=1$, it is necessary to assume that $\varepsilon_r^{(\infty)}$ scales as U^3/r. In other words, if the most intense events are to give $h=1$, then the underlying physics of these events should have nothing to do with 1941 Kolmogorov scaling. (Recall that I have assumed that all moments exist, and that they behave as power laws in r. It is reasonable to worry if this could occur in the absence of residual effects of 1941 Kolmogorov scaling.) On the other hand, if $h=2/3$ as in She-Leveque, then a residual effect of 1941 Kolmogorov scaling is assumed present even for the most intense events. Thus it is not surprising that $h=2/3$ leads to some remnant of the linear scaling of $\zeta(p)$ with p at large p, but that $h=1$ eliminates the linear term completely. It is also not surprising that $h=2/3$ leads to a gap in an underlying PDF since scaling ideas are used to exclude the most intense events allowed by the underlying probabilistic description. At the present time I do not know whether either of these two pictures is physically correct, but it is clear that they differ in an underlying assumption about the dynamics of the most intense events, and not just in the formal structure of the probabilistic description. Despite this basis physical difference, however, the two models cannot be distinguished experimentally at the level of scaling exponents.

Note added. Since the first version of this paper was written, some new information has become available which pertains to the above discussion, but does not resolve the underlying controversy. Ruiz Chavarria *et al.* [9] have experimentally studied the hierarchy of moments underlying the She-Leveque model, and have found strong support for the underlying hierarchical structure. Chen and Cao [10] have pointed out, however, that the hierarchical model of She and Leveque does not exclude the possibility that $h=1$. Pedrizzetti *et al.* [11] have measured the PDF of the ratio variable $\varepsilon_r/\varepsilon_{br}$ with $b<1$, and have found no evidence for a gap. This suggests that $h=1$, but She [12] cautions that the scaling exponents for the ratio variable and for the averaged dissipation ε_r need not be the same. The question raised in this paper remains unresolved, and is of continued interest.

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